Pseudouptriangular Decomposition Method for Constrained Multibody Systems Using Kane's Equations

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A pseudouptriangular decomposition method used to obtain an orthogonal complement array to automatically reduce the equations of motion is presented. The method is based on a successive multiplication of Householder transformations to develop an uptriangular matrix whose column vectors are orthogonal to the rows of the corresponding constraint matrix. The columns of the transformation matrix represent the directions for which the system is unrestrained in n-dimensional space. The automatic generation of the transformation matrix makes the analysis of constrained multibody systems more tractable. The method developed is found to be useful when the equations of motion are developed using Kane's equations with undetermined multipliers. A discussion of the advantages of the method presented vs previous methods is presented together with illustrative examples.

I. Introduction

THE development of generic computer programs capable of performing dynamic analysis of constrained multibody systems has been the interest of many dynamicists for the past two decades. The advantages to the methods employed have been the result of a number of previous research publications.[1-4] It is found that the use of Kane's equations leads to efficient algorithms that are tailored for the development of computer programs that can handle a large class of mechanical systems.

The equations of motion, together with a set of nonlinear algebraic constraint equations, form the governing equations of motion. In order to improve computing efficiency and express the equations of motion in terms of independent coordinates, it is desirable to find algorithms that automatically reduce the equations of motion and hence will lead to the natural dynamical equations of the system. Methods that lead to such an approach have been included in the singular value decomposition method,[6] the use of the zero eigenvalue theorem,[5] and the generalized coordinate partitioning method[7] among others. In Ref. 5, Kamman and Huston used a theorem introduced by Walton and Steeves[8] called the zero eigenvalue theorem to generate an orthogonal matrix. The method works as follows for a dynamical system of n generalized coordinates and m constraints: A square matrix given by B^TB is formed, where B denotes the $m \times n$ constraint Jacobian matrix that is assumed to be of a full rank. An orthogonal matrix C of independent eigenvectors corresponding to the zero eigenvalues is next calculated. The complement array C is then used to reduce the governing equations of motion. A similar approach presented by Singh and Likins[5] uses the singular value decomposition to extract the complement orthogonal matrix C. Solving for the eigenvalue problem in both cases results in computational time deficiency, especially if the structure size becomes more pronounced. Both Kamman and Huston's and Singh and Likins' formulation of the equations of motion are derived using Kane's equations, and therefore, their methods proved to be suitable when the equations of motion are presented in their developed forms. Wehage and Haug[7] developed an algorithm to identify independent generalized coordinates by using LU factorization of the constraint Jacobian matrix (B). This method occasionally leads to poorly conditioned matrices. When this occurs, it is necessary to choose a new set of independent coordinates by repeating the LU factorization process. This procedure not only increases the computing time, but also propagates integration errors. Another disadvantage is in the formulation of equations of motion. Other researchers have established different criteria for generating the orthogonal matrix C without solving for the eigenvalue problem.[9]

In this paper, a new method is introduced that extracts an orthogonal matrix based on a simple successive multiplication of Householder transformations. The advantage of this approach stems from avoiding the solution for an eigenvalue problem that is computationally expensive. This approach is also found to be extremely useful for large-scale systems when the equations of motion are formulated using Kane's equations with undetermined multipliers. This approach is far-reaching in its applications, and it leads to algorithms that can be used to evaluate the constraining forces and torques.

This paper is divided into five sections. In the first section, Kane's equations with undetermined multipliers are introduced, the pseudouptriangular decomposition method derivation forms the second section, the third section represents the automatic reduction and solution of the governing equations of motion. Two examples followed by the conclusion form the last two sections.

II. Kane's Equations with Undetermined Multipliers

In previous multibody dynamics research on constrained systems, Huston and Passerello,[2] Kamman and Huston,[5] Singh and Likins,[6] and Wampler, Buffinton, and Shu-Hui[4] all showed how Kane's equations could be used for systems subject to additional constraints without ever introducing the undetermined multipliers as in the Lagrange method. This is only true in the case where the constraining forces and torques are of no interest in the analysis. Introducing the undetermined multipliers in Kane's equations will make the analysis more complete in the case where the generalized constraint forces need to be computed and monitored. In applications like biodynamics, robotics, and space structure mechanisms, the evaluation of the constraint forces and torques generated by imposing on a system a set of constraint equations is, in certain cases, very useful. In this paper, we will illustrate how we can introduce the undetermined multipliers in Kane's equations and still take advantage of the computational efficiency

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when the dynamical equations are given without the undetermined multipliers; hence, the method introduced in this paper becomes simply an alternative to previously developed methods. However, if the undetermined multipliers need to be computed, the simple relationship between the modified Kane's equations and Kane's equations with the undetermined multipliers could be used to evaluate the constraining forces and torques. An illustration of Kane's method is as follows: Consider a multibody system that consists of N bodies as depicted in Fig. 1. Let this system be subjected to an externally applied force field and a constraint force field. Let the externally applied force field applied to a particular body $\boldsymbol{B}_k$ be replaced by a single force $\boldsymbol{F}_k$ passing through the mass center G_k of $\boldsymbol{B}_k$, and a moment $\boldsymbol{M}_k$ about G_k. From the same analogy, let the constraint force field be also transformed into a constraint force $\boldsymbol{F}_{ck}$ and a moment $\boldsymbol{M}_{ck}$ about the mass center G_k. The equations of motion of a multibody system could be obtained using Kane's dynamical equations as

$$f_\ell + f_\ell^* = 0 \qquad (\ell = 1, \ldots, 6N) \tag{1}$$

where f_ℓ denotes the generalized active forces and f_ℓ^* are the generalized inertia forces. Following the notation used by Huston et al.,[1,2] those forces are found to be

$$f_\ell = F_{km} V_{k\ell m} + M_{km} \omega_{k\ell m} \tag{2}$$

and

$$f_\ell^* = F_{km}^* V_{k\ell m} + M_{km}^* \omega_{k\ell m} \tag{3}$$

where $V_{k\ell m}$ and $\omega_{k\ell m}$ represent the partial velocity and partial angular velocity arrays, respectively, F_{km}^* and M_{km}^* are the inertia forces and torques, and F_{km} and M_{km} are the components of forces and moments of the external forces acting on body K. It should also be noted that in both Eqs. (2) and (3) all the parameters on the right-hand side of the equations are expressed with respect to a set of mutually perpendicular unit vectors fixed in a Newtonian inertial space R. The indices K represent the body number and go from 1 to N, ℓ is associated with the generalized coordinates and goes from 1 to $6N$, and m varies from 1 to 3 and is associated with the unit vectors fixed in $R(\boldsymbol{n}_{om})$. For an open chain, the generalized inertia forces could be expressed in explicit form suitable for computer implementation as

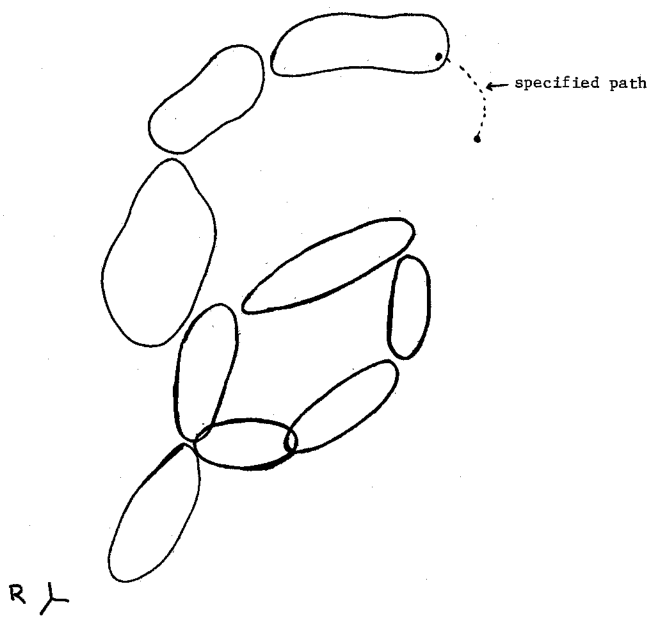

Fig. 1 A general multibody system.

$$a_{\ell p} \ddot{x}_p + b_{\ell p} \dot{x}_p + C_{\ell pn} \dot{x}_p \dot{x}_n = -f_\ell \tag{4}$$

where the coefficients $a_{\ell p}$, $b_{\ell p}$, and $c_{\ell pn}$ are given by

$$a_{\ell p} = m_k V_{k\ell m}^T V_{k\ell m} + \omega_{k\ell m}^T I_{k\ell m} \omega_{knp} \tag{5}$$

$$b_{\ell p} = m_k V_{k\ell m}^T \dot{V}_{k\ell m} + \omega_{k\ell m}^T I_{kmn} \dot{\omega}_{knp} \tag{6}$$

$$c_{\ell np} = -\omega_{k\ell m}^T e_{mjs} \omega_{ksn} I_{kjp} \omega_{qp} \tag{7}$$

Note that the partial velocities and partial angular velocities and their respective rates of change are block matrices that are easily coded for computers; e_{mjs} is the permutation symbol and I_{kmn} represents the inertia dyadic of body k. The readers are urged to review Refs. 2 and 3 for further details on the formulation of Eqs. (5–7). To extend Kane's equations to include the undetermined multipliers as in the Lagrance method, it is shown in Ref. 12, that Kane's equations are equivalent to Lagrange equations; hence,

$$-f_\ell^* = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_\ell}\right) - \frac{\partial T}{\partial x_\ell} = f_\ell \qquad (\ell = 1, \ldots, 6N) \tag{8}$$

In Eq. (8), T denotes the kinetic energy of the conservative system of the N bodies given in the space provided by R, and x_ℓ are the generalized coordinates. Using the principle of virtual work, we could write Eq. (8) as

$$\delta W = \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_\ell}\right) - \partial T/\partial x_\ell - f_\ell\right] \delta x_\ell = 0 \tag{9}$$

The foregoing developed Lagrange equations of motion can be extended if the system of the multibody system is subjected to $m\,(m < 6N)$ constraint equations given by

$$\phi_i(x_\ell, t) = 0 \qquad (i = 1, 2, \ldots, m) \tag{10}$$

Employing the method of Lagrange multipliers, we can modify Eq. (9) to include the constraints as

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_\ell}\right) - \frac{\partial T}{\partial x_\ell} - f_\ell - \lambda_i B_{i\ell}\right] \delta x = 0 \tag{11}$$

where λ_i are the undetermined mulitpliers and $B_{i\ell}$ represent the Jacobian matrix obtained by differentiating Eq. (10). Since x_ℓ is arbitrary without violating the constraints, Eq. (11) can be written as

$$f_\ell + f_\ell^* + B_{\ell i} \lambda_i = 0 \tag{12}$$

Equation (12) is therefore the extended Kane's equations with the undetermined multipliers and, hence, should be called "Kane's equations with the undetermined multipliers." The dynamical equations of motion are then given by Eq. (12) together with Eq. (10) or more conveniently by their derivatives. These equations are for both holonomic and simple nonholonomic systems. It is important to note at this stage that Eq. (12) has the same advantages as Eq. (1) in which the "nonworking" forces are automatically eliminated and the use of the generalized speeds allows one to select them not only as the time derivatives of the generalized coordinates as presented in previous analysis, but as convenient linear combinations of them as well. This will allow Eq. (1) or (12) to be represented in the simplest form.

The objective at this point is to show how a system as given by Eq. (12) or, for that matter, the one given by Eq. (1) (if there is no need to evaluate the undetermined multipliers) can be solved, if it is subjected to m independent constraints. It is obvious that the number of degrees of freedom must reduce from $6N$ to $6N$-m. One method used to reduce automatically the equations of motion is next presented.

III. Pseudouptriangular Decomposition (PUTD) Theorem

The pseudouptriangular decomposition theorem utilizes the Householder transformation to formulate an uptriangular matrix whose column vectors are linearly independent. In order to illustrate the methodology used, let us review some fundamental concepts related to the Householder transformation matrix. Given two vectors x and $y \in R^n$, where R^n denotes an n-dimensional real space, the Householder transformation states that there exists a unit vector u such that

$$Hx = \sigma y \tag{13}$$

where the Householder transformation matrix is given by

$$H = I - 2uu^T \tag{14}$$

In Eq. (14), I is an $n \times n$ identity matrix, whereas σ in Eq. (13) is a scalar quantity found to be

$$\sigma = \sqrt{\frac{x^T x}{y^T y}} \tag{15}$$

and u^T, x^T, and y^T are the transposes of u, x, and y respectively. It is also known from the Householder transformation that if $x \neq \sigma y \neq 0$, then u is uniquely defined as

$$u = \frac{\sigma y - x}{\sqrt{(\sigma y - x)^T (\sigma y - x)}} \tag{16}$$

If we let σy be simply w, then Eq. (13) could be expressed as

$$Hx = w \tag{17}$$

where w gives the same euclidean norm as that of x; it can also be shown that Eq. (16) reduces to

$$u = \frac{w - x}{\sqrt{(w - x)^T (w - x)}} \tag{18}$$

It can be then illustrated that $H^T H = I$; therefore, $H^T = H^{-1}$, which implies that the Householder transformation matrix is an orthonormal matrix that gives a full-rank transformation. The steps next presented will illustrate how the pseudouptriangular decomposition is formulated.

Step 1

Consider a matrix $B_{\bar{q}}$ of order $(n \times m)$ that has m linearly independent vectors B_i; in this case, they are the column vectors of the matrix $B_{\bar{q}}$. In our analysis of multibody dynamics, $B_{\bar{q}}$ will represent the Jacobian matrix obtained by differentiating Eq. (10). Let $B_{\bar{q}}^{(1)}$ represent the initial matrix where the superscript 1 denotes the first iteration, so

$$B_{\bar{q}}^{(1)} = B_{\bar{q}} \tag{19}$$

or

$$B_{\bar{q}}^{(1)} = [B_1^{(1)}, B_2^{(1)}, \ldots, B_m^{(1)}] \tag{20}$$

$$x^{(1)} = B_1^{(1)} \tag{21}$$

and let us choose a vector $w^{(1)}$ to be linearly independent of $x^{(1)}$ such that Eq. (17) applies. We can easily show that $w^{(1)}$ could be chosen to be a vector such that its first element is given by the norm of $x^{(1)}$, and the rest of the vector elements are zero. So let

$$w^{(1)} = \begin{Bmatrix} \|B_1^{(1)}\| \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{22}$$

Note that the choice of $w^{(1)}$ is not unique: however, it will become clear when the pseudouptriangular decomposition is achieved that our choice of w makes it a lot simpler for the analysis to obtain an orthogonal basis for the column vectors of the pseudouptriangular decomposition. Substituting Eqs. (21) and (22) into Eq. (18), the Householder transformation could be evaluated using Eq. (14). In our first iteration, let us denote by $H_{ij}^{(1)}$ the Householder transformation relating $x^{(1)}$ and $w^{(1)}$.

Step 2

Using $H_{ij}^{(1)}$ from step 1, we can write the following:

$$B_{\bar{q}}^{(2)} = H_{ij}^{(1)} B_{ji}^{(1)} \tag{23}$$

where $B_{\bar{q}}^{(2)}$ represents a new set of linearly independent vectors and the superscript (2) denotes the second iteration. It can be shown that $B_{\bar{q}}^{(1)}$ has the following form:

$$B_{\bar{q}}^{(2)} = [w^{(1)}, B_2^{(2)}, B_3^{(2)}, \ldots, B_m^{(2)}] \tag{24}$$

with $B_i^{(2)}$ $(i = 1,2,\ldots,m)$ being the new column vectors of $B_{\bar{q}}^{(2)}$. We set $x^{(2)}$ to be $B_2^{(2)}$ and define its linearly independent vector $w^{(2)}$ as

$$w^{(2)} = \begin{Bmatrix} B_{1,2}^{(2)} \\ [\|B_2^{(2)}\|^2 - (B_{1,2})^2]^{1/2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{25}$$

Knowing $x^{(2)}$ and $w^{(2)}$, we can obtain the Householder transformation that relates the two vectors. Let us define such a transformation by $H^{(2)}$, which, in turn, can be used in the third step to obtain $B_{\bar{q}}^{(3)}$. Repeating the procedures up to now $(q < m)$, we obtain step q.

Step q

At this step, we allow $x^{(q)}$ to be given by the q column vector $B^{(q)}$ of the matrix $B_{\bar{q}}^{(q)}$ and express $w^{(q)}$ a

$$w^{(q)} = \begin{Bmatrix} B_{1,q}^{(q)} \\ \cdot \\ \cdot \\ \cdot \\ B_{q-1,q}^{(q)} \\ \left[\|B_q^{(q)}\|^2 - \sum_{r=1}^{q-1} (B_{r,q}^{(q)})^2\right]^{1/2} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{26}$$

It should be noted that Eq. (26) is a general representation of the selection of the w vectors at any step q. At the qth step $(1<q<m)$, the Householder transformation $H^{(q)}$ is then evaluated, and by premultiplying it by the matrix $B_q^{(q)}$, we obtain new matrix $B_q^{(q+1)}$, which is composed of column vectors that are linearly independent. In matrix form, it can be written as

$$B^{(q+1)}=H^{(q)}B^{(q)} \tag{27}$$

where

$$B^{(q)}=H^{(q-1)}B^{(q-1)} \tag{28}$$

or

$$B^{(q)}=H^{(q-1)}H^{(q-2)},\ldots,H^{1}B^{(1)} \tag{29}$$

An explicit form of $B^{(q+1)}$ in terms of its column vectors is

$$B^{(q+1)}=[w^{(1)},w^{(2)},\ldots,w^{(q)},B^{(q+1)},$$
$$B^{(q+1)},\ldots,B_m^{(q+1)}] \tag{30}$$

To accomplish the pseudouptriangular decomposition, we continue updating the B matrix until all the column vectors will be given by the w vectors. We finally obtain

$$B^{(m+1)}=H^{(m)}B^{(m)} \tag{31}$$

where

$$B^{m}=H^{(m-1)}H^{(m-2)},\ldots,H^{(1)}B^{(1)} \tag{32}$$

Substituting Eq. (32) into (31) and expressing it in compact form lead to

$$B^{(m+1)}=HB \tag{33}$$

where

$$H=H^{(m)}H^{(m-1)},\ldots,H^{(1)} \tag{34}$$

$B^{(m+1)}$ gives an uptriangular matrix composed of the w vectors as

$$B^{(m+1)}=[w^{(1)},w^{(2)},w^{(3)},\ldots,w^{(m)}] \tag{35}$$

If we substitute the elements of vector array w^{i} $(i=1,\ldots,m)$ into Eq. (35), $B^{(m+1)}$ takes the following particular form:

$$B^{(m+1)}=\begin{bmatrix} * & * & . & . & . & * \\ 0 & * & . & . & . & * \\ 0 & 0 & . & . & . & * \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ .. & . & . & . & . & \vdots \\ . & . & . & . & . & * \\ 0 & 0 & . & . & . & 0 \end{bmatrix} \tag{36}$$

Equation (34) could also be expressed as

$$H\triangleq\prod_{i=1}^{m}(I-2u^{(i)}u^{(i)T}) \tag{37}$$

where Π denotes the product sum for i varying from 1 to m. From Eq. (35), we can see that the column vectors of the matrix $B^{(m+1)}$ are linearly independent; hence, an orthogonal basis for the column vectors could be obtained through the Gramm-Schmidt process. With the Gramm-Schmidt process, an orthonormal set of vectors d_i $(i=1,2,\ldots,m)$ can be introduced such that

$$d_1=w^{(1)}/w^{(1)} \tag{38}$$

and

$$d_i=[w^{(i)}-d_j w^{(i)} d_j]/(w^{(i)}-d_j w^{(i)} d_j)$$
$$(i=2,3,\ldots,m)$$
$$(j=i-1) \tag{39}$$

Using Eqs. (38) and (39), we can form a matrix D_1 whose column vectors span the same basis as the vectors given by $B^{(m+1)}$. D_1 is a matrix of order $(6N\times m)$ whose column vectors are given by the Gramm-Schmidt process as

$$D_1=[d_1,d_2,d_3,\ldots,d_n] \tag{40}$$

More explicitly, D_1 could be expressed as

$$D_1=\left[\frac{I_{(m\times m)}}{0_{(6N-m)\times m}}\right] \tag{41}$$

where I denotes an identity matrix and 0 a zero matrix. By deduction, it can be said that a matrix D_2 can be formulated such that its column vectors are orthogonal to D_1 and, hence, orthogonal to $B^{(m+1)}$. D_2 can then simply be defined as

$$D_2=\left[\frac{0_{(6N-m)\times(6N-m)}}{I_{(6N-m)\times(6N-m)}}\right] \tag{42}$$

From Eqs. (27), (41), and (42), we can write

$$BH^{T}D_2=0 \tag{43}$$

where $H^{T}D_2$ forms the basis for the null space of B. So if the constraint equations are given by

$$By=0 \tag{44}$$

then

$$BH^{T}D_2q=0 \tag{45}$$

is a solution to Eq. (44) in which q is an arbitrary column vector representing the new generalized speeds related to the physically meaningful generalized speeds y.

Theorem. Consider any set of m independent linear homogeneous equations in n unknowns, where $n>m$ is given by

$$By=0 \tag{46}$$

There exists an orthogonal matrix equal to $H^{T}D_2$ such that $BH^{T}D_2=0$, where H is found through a successive multiplication of Householder's transformations such that HB is an uptriangular matrix and D_2 is a matrix whose column vectors are orthogonal to HB. The general solution to Eq. (46) is also given by

$$y=H^{T}D_2q \tag{47}$$

where q is an arbitrary column vector of order $n-m$.

IV. Automatic Reduction of the Equations of Motion and Solution Algorithm

Let us refer to the reduced equations of motion as the "natural dynamics equations" that will consist of $6N-m$ degrees of freedom. The constrained multibody system will therefore be free to move along the $(n-m)$ unrestrained vectors. To illustrate the automatic reduction of the equations of motion of a multibody system when subjected to m additional constraint equations, we take Eq. (12) and premultiply it by an orthogonal complement array given by $H^T D_2$ that yields the following:

$$K_p + K_p^* = 0 \tag{48}$$

where

$$K_p = H^T D_2 f_\ell \tag{49}$$

and

$$K_p^* = H^T D_2 f_\ell^* \tag{50}$$

Equation (48) represents the modified Kane's equations for the constrained system. An explicit form of these equations could be found using the resulting equations given by Eq. (4) that yields

$$A\ddot{x} + B\dot{x} + C\dot{x}\dot{x} = -K_p \tag{51}$$

where the coefficients A, B, and C are given by

$$A = H^T D_2 a$$
$$B = H^T D_2 b$$
$$C = H^T D_2 c \tag{52}$$

Equation (51) together with Eq. (44) forms a set of $(6N)$ differential and algebraic equations that, once integrated, would lead to the time history of the constrained multibody system dynamics. In representing the results as a set of differential equations ready for integration, the differentiable form of Eq. (44) together with Eq. (51) could be expressed as

$$G_{r\ell}\dot{Y}_\ell = R_r \qquad (r,\ell = 1,\ldots,6N) \tag{53}$$

where G and R could be given in a partitioned matrix form as

$$G = \left[\frac{A}{B}\right] \tag{54}$$

and

$$R = \left[\frac{-K_p - BY - CYY}{-\dot{B}Y}\right] \tag{55}$$

where Y denotes the generalized speed that, in this case, are the derivative of the generalized coordinates x. It should be noted that the generalized speeds can always be chosen such that they are a linear combination of the x derivatives and time, which will allow the equations of motion to be represented as Eq. (53). To solve numerically for the $6N$ differential equations, G is assured to be a square matrix (nonsingular), and hence, its inverse exists.

Illustrative Example

To illustrate the foregoing procedures as well as their relationship to methods developed in Refs. 4 and 5, we selected the example given by Wampler, Buffinton, and Shu-Hui. Consider the mechanism shown in Fig. 2. The system consists of two subsystems, the first of which is composed solely of a particle P_1 of mass M that slides on a frictionless track T so that its position may be described by the single generalized coordinate q_1. The second subsystem consists of two rigid massless links, link 1 and link 2, each of length L, and two particles P_2 and P_3, each of mass m, assembled as shown. Motors (not shown) exert a torque R on link 1 and a torque S between link 1 and link 2. The equations of motion could easily be obtained using Kane's equations and setting the generalized coordinates to be q_1, q_2, and q_3.

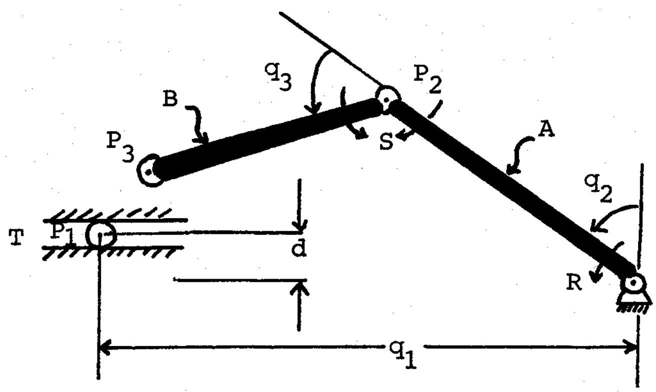

Fig. 2 Unconstrained system.

$$-M\ddot{q}_1 = 0$$
$$-mL^2\{(3+2C_3)\ddot{q}_2 + (1+c_3)\ddot{q}_3 + s_3[q_2^2$$
$$-(q_2+q_3)]\} = R + mgL(2s_2 + s_{23})$$
$$-\mathrm{m}L^2[(1+c_3)\ddot{q}_2 + \ddot{q}_3 + S_3\dot{q}_2^2] = S + mgLS_3 \tag{56}$$

Let the generalized speeds be conveniently defined

$$u_1 \triangleq \dot{q}_1, \qquad u_2 \triangleq \dot{q}_2, \qquad u_3 \triangleq \dot{q}_3$$

The equations of motion could then be expressed as

$$-M\dot{u}_1 = 0$$
$$R + mgL(2S_2 + S_{23}) - mL^2(3+2c_3)\dot{u}_2$$
$$+(1+c_3)\dot{u}_3 + s_3[u_2^2 - (\mathrm{u}_2 + u_3)^2] = 0$$
$$S + mgLS_{23} - mL^2(1+c_3)\dot{u}_2 + \dot{u}_3 + S_3 u_2^2 = 0 \tag{57}$$

where g is the acceleration of gravity, $S_i \triangleq \sin q_i$, $c_i \triangleq \cos q_i$ $(i=2,\ 3)$, and $S_{23} \triangleq \sin(q_2+q_3)$. Consider next the system given in Fig. 3 where a constraint on P_3 is imposed such that it coincides with P_1. The constraints are obtained by equating the velocity of P_1 and P_2, which yields

$$B_{i\ell}u_\ell = 0 \qquad (\ell = 1,2,3 \text{ and } i = 1,2) \tag{58}$$

where

$$B_{i\ell} = \begin{bmatrix} S_2 & LS_3 & LS_3 \\ c_2 & -\mathrm{L}(1+c_3) & -Lc_3 \end{bmatrix} \tag{59}$$

Since the system now possesses one degree of freedom utilizing the method proposed in this paper, we should be able to reduce Eq. (57) by embedding the constraints given by Eq. (58). It will be better to use numerical values for Eq. (59) so we can illustrate all the steps of the pseudouptriangular decom-

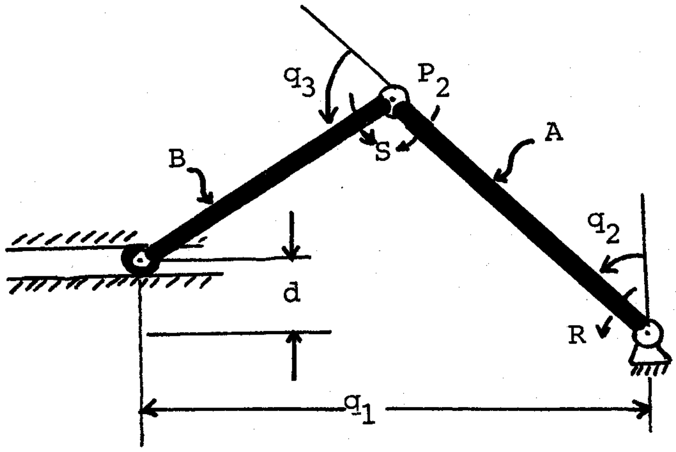

Fig. 3 Constrained system.

position. Let us assume that $q_2=30$ deg, $q_3=60$ deg, and $L=1$. Hence, the matrix $B_{i\ell}$ becomes

$$B_{i\ell}=\begin{bmatrix} 0.5 & 0.866 & 0.866 \\ 0.866 & -1.5 & -0.5 \end{bmatrix} \tag{60}$$

In step 1 of the transformation, Eq. (21) gives

$$x^{(1)}=(0.5 \quad 0.866 \quad 0.866) \tag{61}$$

Using Eq. (22), we obtain

$$w^{(1)}=(1.3229 \quad 0.0 \quad 0.0) \tag{62}$$

Equation (14) then gives the Householder transformation $H^{(1)}$ as

$$H^{(1)}=\begin{bmatrix} 0.3780 & 0.6547 & 0.6547 \\ 0.6547 & 0.3110 & -0.6940 \\ 0.6547 & -0.6890 & 0.3110 \end{bmatrix} \tag{63}$$

Using Eq. (23) together with the results of Eq. (63), we obtain $B^{(2)}$. Performing step 2 yields the transformation matrix $H^{(2)}$ found to be

$$H^{(2)}=\begin{bmatrix} 1.000 & 0.0 & 0.0 \\ 0.0 & 0.2943 & 0.9557 \\ 0.0 & 0.9557 & -0.2943 \end{bmatrix} \tag{64}$$

From Eqs. (34) and (42), we obtain the desired transformation matrix H^TD_2 that will be used to reduce the governing equations of motion. The latter is found to be

$$H^TD_2=\begin{bmatrix} 0.4330 \\ 0.5000 \\ -0.7500 \end{bmatrix} \tag{65}$$

It can easily be shown that Eq. (43) is equal to zero after substitution. What is important at this stage is to compare our method with the previous ones and especially with the methodologies employed in Ref. 4 by Wamper et al., and with the one used by Kamman and Huston in Ref. 5 that utilizes the zero eigenvalue theorem. In the first paper, the authors stated that if a system of N particles is subjected to m additional constraints, then the generalized speeds of the system denoted by u are given by

$$u=\underset{\sim}{\alpha}\bar{u}+\underset{\sim}{\beta} \tag{66}$$

such that $\underset{\sim}{\alpha}$ is an orthogonal complement array used to reduce the equations of motions and $\bar{u}$ are the new generalized speeds. If the constraints are holonomic, then $\underset{\sim}{\beta}$ is equal to zero. Obtaining $\underset{\sim}{\alpha}$ is what the method proposed in this paper does, as well as the singular value decomposition and the zero eigenvalue theorem. Since the choice of $\underset{\sim}{\alpha}$ is not unique, what is left to demonstrate is how efficient all these procedures are, and which method has the computational advantages to determine $\underset{\sim}{\alpha}$. The method proposed in Ref. 4 works as follows. Let the constraints be given by

$$Bu+g=0 \tag{67}$$

where B is an $m\times n$ matrix and g an $m\times 1$ matrix. The new generalized speeds can be defined as

$$\bar{u}=Ru+Q \tag{68}$$

where R and Q are functions of the generalized coordinates and time. The choice for R and Q is arbitrary so long as R is such that the composite system of Eqs. (67) and (68) holds. Solving for the composite system to be more precise, the solutions to Eq. (24) of Ref. 4 yields a solution of the form of Eq. (66) where $\underset{\sim}{\alpha}$ is given by

$$\underset{\sim}{\alpha}=E^{-1}\left[\frac{0}{(n^{I}m\times n)}\right] \tag{69}$$

where

$$E=\begin{bmatrix} \dfrac{\begin{matrix} B \\ (m\times n)\end{matrix}}{\begin{matrix} R \\ (n-m\times n)\end{matrix}} \end{bmatrix} \tag{70}$$

Two things need to be highlighted about the foregoing analysis: First, the selection of R is not trivial (the authors showed how for certain special cases R is selected), and second, finding the inverse of the matrix E could lead to some serious problems when E is close to being singular. In any case, $\underset{\sim}{\alpha}$ is equivalent to H^TD_2 developed by the method proposed in this paper. In Ref. 4, Eq. (48) gives the $\underset{\sim}{\alpha}$ to the illustrative example just described as

$$\underset{\sim}{\alpha}=\begin{bmatrix} 1.0 \\ 1.1547 \\ -1.7320 \end{bmatrix} \tag{71}$$

that can be written as

$$\underset{\sim}{\alpha}=2.309\begin{bmatrix} 0.4330 \\ 0.5000 \\ -0.7500 \end{bmatrix} \tag{72}$$

Comparing Eq. (72) to Eq. (65), we see that $\underset{\sim}{\alpha}$ is simply H^TD_2 multiplied by a constant. Another method that extracts the orthogonal complement array H^TD_2 or $\underset{\sim}{\alpha}$ is the zero eigenvalue theorem as exposited by Kamman and Huston in Ref. 5. A brief illustration of the theorem is as follows: Consider the B matrix given by Eq. (46); then, we define a matrix S such that

$$S=B^TB \tag{73}$$

where B^T is the transpose of B, and S is a symmetric square matrix. Therefore, there exists an orthogonal matrix T such that

$$T^TST=\Lambda \tag{74}$$

where Λ is an $n\times n$ diagonal matrix with real elements or eigenvalues, λ^i $(i=1,2,\ldots,n)$. These eigenvalues are nonnegative and it is also seen that

$$u=T\bar{u} \tag{75}$$

where T is a matrix whose columns are the eigenvectors corresponding to the zero eigenvalues. Further details on the method can be found in Refs. 5 and 8. It should be noted that both the pseudouptriangular decomposition and the zero eigenvalue theroem are both "algorithmic" techniques that are easily coded for numerial computer solutions. The advantage of using the pseudouptriangular decomposition is the fact that it is strictly based on matrix multiplication operation; there is no diagonalization, eigenvalue determination, and matrix inversions involved.

Comparative Simulation of a Planar Triple Pendulum

For a simple example illustrating some of these ideas, consider the planar triple pendulum shown in Fig. 4a. The three rods are identical, having length ℓ, and there are frictionless pins at the joints: 0_1, 0_2, and 0_3. The system has three degrees of freedom that may be described by the orientation angles θ_1, θ_2, and θ_3 shown in the figure. If we use Kane's equations, the equations of motion of the sytem take the form

$$\begin{aligned}&(23+18C_2+6C_3+6C_{2+3})\ddot{\theta}_1+(10+9C_2=6C_3+3C_{2+3})\ddot{\theta}_2\\&+(2+3C_3+3C_{2\times 3})\ddot{\theta}_3=-(g/\ell)(15S_1+9S_1+9S_{1+2}+3S_{1+2+3})\\&-(9S_2+3S_{2+3})\dot{\theta}_1^2+(9S_2-3S_3)(\dot{\theta}_1+\dot{\theta}_2)^2\\&+(3S_{2+3}+3S_3)(\dot{\theta}_1+\dot{\theta}_2+\theta_3)^2\end{aligned} \tag{76}$$

$$\begin{aligned}&(10+9C_2+6C_3+3C_{2+3})\ddot{\theta}_1+(10+6C_3)\ddot{\theta}_2+(2+3C_3)\ddot{\theta}_3\\&=-(g/\ell)(9S_{1+2}+3S_{1+2+3})-9S_2+(3S_{2+3})\dot{\theta}_1^2\\&-3S_3(\dot{\theta}_1+\dot{\theta}_2)^2+3S_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2\end{aligned} \tag{77}$$

and

$$\begin{aligned}&(2+3C_3+3C_{2+3})\ddot{\theta}_1+(2+3C_3)\ddot{\theta}_2+2\ddot{\theta}_3\\&=3(g/\ell)S_{1+2+3}-3S_{2+3}\dot{\theta}_1^2-3S_3(\dot{\theta}_1+\dot{\theta}_2)^2\end{aligned} \tag{78}$$

where $C_i=\cos\theta_i$, $C_{i+j}=\cos(\theta_i+\theta_j)$, etc.

A "closed loop" or constraint may be formed by fixing the end point P of the pendulum. Hence, let P be fixed at a point P_0 having coordinates (a, b) relative to the X-Y coordinate system shown in Fig. 4b. This constrained system has only one degree of freedom. Two scalar constraint equations relating the coordinates θ_1, θ_2, and θ_3 may be obtained from the position vector equation

$$0_10_2+0_20_3+0_3P_0+P_00_1=0 \tag{79}$$

That is, considering the horizontal and vertical components of this equation leads to the equations

$$\begin{aligned}&S_1+S_{1+2}+S_{1+2+3}=a/\ell\\&C_1+C_{1+2}+C_{1+2+3}=b/\ell\end{aligned} \tag{80}$$

that, upon differentiation, becomes

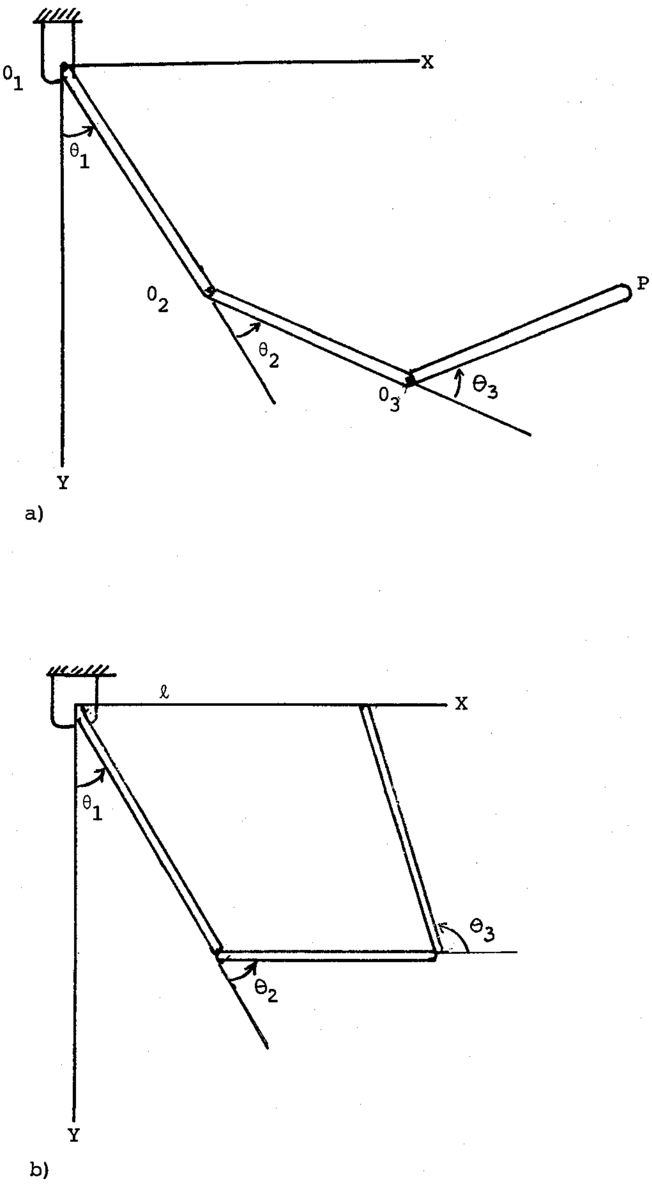

Fig. 4 Planar triple pendulum: a) Unconstrained. b) Constrained

$$\begin{aligned}&(C_1+C_{1+2}+C_{1+2+3})\dot{\theta}_1\\&\quad+(C_{1+2}+C_{1+2+3})\dot{\theta}_2+C_{1+2+3}\dot{\theta}_3=0\\&(S_1+S_{1+2}+S_{1+2+3})\dot{\theta}_1+(S_{1+2}+S_{1+2+3})\dot{\theta}_2\\&\quad+S_{1+2+3}\dot{\theta}_3=0\end{aligned} \tag{81}$$

Equations (81) represent the constraint equations as described by Eq. (49).

To simplify this analysis, let P be fixed on the X axis at a point P_0, a distance ℓ from 0_1. The system then takes the form of a rhombic linkage as shown in Fig. 4b. In this case, $a=\ell$, $b=0$, and the constraint equations (81) are seen to be satisfied by the relations $\theta_2=\pi/2-\theta_1$ and $\theta_3=\pi/2+\theta_1$. The coefficient matrix B of Eqs. (80) and (81) may then be expressed as

$$B=\begin{bmatrix}0 & -C_1 & -C_1\\ 1 & (1-S_1) & -S_1\end{bmatrix} \tag{82}$$

The governing equation of the rhombic linkage of Fig. 4b could easily be found to be the solution of the pendulum equation.

$$\ddot{\theta}_1 + (6g/5\ell)\sin\theta_1 = 0 \tag{83}$$

A numerical integration of the equation of motion was performed in accordance with the method presented in this paper and results were checked against the solution given by Eq. (83). Tables 1 and 2 provide the numerical solutions. An accuracy of up to four digits was achieved.

V. Conclusion

An alternative approach to the dynamics of constrained multibody systems is presented. Unlike the singular value decomposition method and the zero-eigenvalue theorem used in automatically reducing the equations of motion, the pseudouptriangular decomposition method does not require the solution of an eigenvalue problem. The complement orthogonal transformation matix $H^T D_2$ is obtained through the successive multiplication of Householder transformations. The generalized constraint forces are automatically eliminated by the equations of motion and need not be computed; however, the procedures developed could be used to compute those forces should they become important in the analysis. It is believed that further simulation is needed to judge the efficiency of this method vs other previous algorithms. One of the advantages of the procedures presented in this paper is the use of Kane's equations in which the nonworking constraint forces are eliminated, therefore reducing enormously the computational burden.

Table 1 Results from the numerical integration of Eq. (83)

Time (s)	θ (rad)	$\dot{\theta}$ (m/s)	$\ddot{\theta}$(m/s^2)
0.1	0.3095	−0.7758	−7.172
0.2	0.1997	−1.380	−4.670
0.3	0.0440	−1.672	−1.037
0.4	−0.1217	−1.578	2.858
0.5	−0.2595	−1.124	6.041
0.6	−0.3381	−0.4178	7.809
0.7	−0.3400	0.3799	7.851
0.8	−0.2649	1.094	6.163
0.9	−0.1293	1.564	3.036
1.0	0.0360	1.676	−0.8461

Table 2 Results from a general-purpose computer program (DYAMUS)

Time	θ	$\dot{\theta}$	$\ddot{\theta}$
0.1	0.3095	−0.7758	−7.172
0.2	0.1997	−1.380	−4.670
0.3	0.0440	−1.672	−1.037
0.4	−0.1217	−1.578	2.858
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References

[1]Huston, R.L. and Passerello, C.E., "On Lagrange Form of D'Alembert's Principle," *The Matrix and Tensor Quarterly*, Vol. 23, 1973, pp. 109–112.

[2]Huston, R.L., Passerello, C.E., and Harlow, M.W., "Dynamics of Multirigid-Body Systems," *Journal of Applied Mechanics*, Vol. 45, 1978, pp. 889–894.

[3]Amirouche, M.L., "Dynamic Analysis of 'Tree-Like' Structures Undergoing Large Motions—A Finite Segment Approach," *Journal of Engineering Analysis-Computational Mechanics Publishing*, Vol. 3, June 1986, pp. 111–117.

[4]Wampler, C., Buffinton, K., and Shu-Hui, J., "Formulation of Equations of Motion for Systems Subject to Constraints," *Journal of Applied Mechanics*, Vol. 52, June 1986, pp. 465–470.

[5]Kamman, J.W. and Huston, R.L., "Dynamics of Constrained Multibody Systems," *Journal of Applied Mechanics*, Vol. 51, Dec. 1984, pp. 899–903.

[6]Singh, R.P. and Likins, P.W., "Singular Value Decomposition for Constrained Dynamic Systems," *Journal of Applied Mechanics*, Vol. 52, Dec. 1985, pp. 943–948.

[7]Wehage, R.A. and Haug, E.J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic System," *ASME Journal of Mechanical Design*, Vol. 104, 1982, pp. 247.

[8]Walton, W.C. Jr. and Steeves, E.C., "A New Matrix Theorem and Its Application for Establishing Independent Coordinates for Complex Dynamical Systems with Constraints," NASA Technical Rept. TR R-326, Oct. 1969.

[9]Hemami, H. and Weimer, F.C., "Modelling of Nonholonamic Dynamic Systems with Applications," *ASME Journal of Applied Mechanics*, Vol. 48, 1981, pp. 117.

[10]Singh, R.P. and Likins, P.W., "Manipulator Interactive Design with Interconnected Flexible Elements," *Proceedings of 1983 Automatic Control Conference*, San Francisco, CA, June 1983, pp. 505–512.

[11]Amirouche, M.L. and Huston, R.L., "Collaborative Techniques in Modal Analysis," *Journal of Guidance, Control, and Dynamics*, Vol. 8, Nov. -Dec. 1985. pp. 782.

[12]Kane, T.R. and Levinson, D.A., *Dynamics: Theory and Applications*, McGraw-Hill, New York, 1985.